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Toric complete intersections and weighted projective space

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Abstract

It has been shown by Batyrev and Borisov that nef partitions of reflexive polyhedra can be used to construct mirror pairs of complete intersection Calabi–Yau manifolds in toric ambient spaces. We construct a number of such spaces and compute their cohomological data. We also discuss the relation of our results to complete intersections in weighted projective spaces and try to recover them as special cases of the toric construction. As compared to hypersurfaces, codimension two more than doubles the number of spectra with $h^{11} = 1$. Altogether we find 87 new (mirror pairs of) Hodge data, mainly with $h^{11} \leq 4$.

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1. Introduction

The first sizeable sets of Calabi–Yau manifolds were constructed as complete intersections (CICY) in products of projective spaces [1,2]. These manifolds have many complex structure deformations but only few Kähler moduli, which are inherited from the ambient space. With the discovery of mirror symmetry [3] the main interest therefore turned to weighted

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projective ($W\mathbb{P}$) spaces, where the resolution of singularities contributes additional Kähler moduli and thus provides a much more symmetric picture [4]. It turned out, however, that mirror symmetry is only approximately realized in this class of models [5,6].

It was then discovered by Batyrev [7] that toric geometry (TG), which generalizes (products of) $W\mathbb{P}$ spaces, provides the appropriate framework for mirror symmetry. In TG the monomial deformations of the hypersurface equations and the gluing data defining the ambient space are given in terms of lattice polytopes that live in a dual pair of lattices. The Calabi–Yau condition for the generic hypersurface requires that these polytopes are dual to one another. This implies, by definition, that the ambient space and the hypersurface are given in terms of a dual pair of reflexive polyhedra, with $\Delta \subset M$ and $\Delta^* \subset N = \mathrm{Hom}(M, \mathbb{Z})$ being exchanged under the mirror involution. Mirror symmetry thus derives from an elementary combinatorial duality.

Because of the large number of hypersurfaces that exist in these spaces [8–11] only little work was directed towards complete intersections. A list of transversal configurations for codimension 2 Calabi–Yau manifolds in $W\mathbb{P}$ spaces was produced by Klemm [12]. As in the case of hypersurfaces, there is, however, in general no mirror construction available in that context [13]. In the toric setup, the mirror construction for hypersurfaces could be extended to general complete intersections by Batyrev and Borisov [14,15]. In addition to a reflexive polyhedron Δ^* that describes the ambient space this involves a decomposition of Δ into a Minkowski sum of polytopes Δ_i that are related to the equations defining the complete intersection. The Calabi–Yau condition implies that these Δ_i are dual to a partition of the vertices of Δ^*, which is called nef because the corresponding divisors are numerically effective. The nef partitions of reflexive polyhedra again feature a beautiful combinatorial duality that implements the mirror involution, as has been proven on the level of Hodge data in [16].

In the present paper we work out a number of examples of toric complete intersection Calabi–Yau manifolds and discuss the relation of this construction to $W\mathbb{P}$ spaces. Identifying CICYs in $W\mathbb{P}$ spaces as a special case of the toric construction will provide, among other benefits, the mirrors for these manifolds. In the case of hypersurfaces in $W\mathbb{P}^4$, the Newton polytope of a transversal quasihomogeneous polynomial [17,18] can be identified with the polyhedron Δ, whose dual provides the toric resolution of the ambient space. It is thus clear that, for codimensions $r > 1$, we should look for the identification by trying to relate the Newton polytopes of the defining polynomial equations of degrees d_i to a nef partition Δ_i of some reflexive polyhedron Δ.

This indeed works for many cases, but the situation is not so straightforward. Already in the case of hypersurfaces reflexivity of the Newton polytope is only guaranteed for dimensions $n \leq 4$ [19] and indeed breaks down for the case of Calabi–Yau 4-folds [20,21]. The most relevant situation from the string theory point of view is that of 3-folds, where already for codimension 2 the Newton polyhedra have dimension 5. Indeed, already in the second example in the list of Ref. [12], namely degree (3, 4) equations in $\mathbb{P}_{1,1,1,1,1,2}$, the Newton polyhedron $\Delta(7)$ for a degree 7 equation, is not reflexive. It is, however, possible to reduce $\Delta(7)$ to a reflexive polyhedron Δ by omitting five points, so that its dual provides a toric resolution of singularities of the weighted projective space. Moreover, in this example, for one of the nef partitions of Δ the Hodge data agree with the $W\mathbb{P}$ result of [12].

In general, $\Delta(d_1+d_2)$ may differ from the Minkowski sum $\Delta(d_1)+\Delta(d_2)$ and neither of the two polytopes has to be reflexive. In many, but not all the cases, we can nevertheless find a simple modification of these polytopes that makes the Hodge data agree, and with some more work one can check the identification in more detail. In the present note we analyze a number of examples from the list in [12] and discuss the different situations that can occur. Apart from our interest in this specific class of examples, we wrote a program that generates all nef partitions with codimension 2 for arbitrary reflexive polyhedra and that computes the Hodge data for the resulting Calabi–Yau manifolds. Using the list of 4-fold polyhedra that were obtained in [21] we produce a sizeable list of Hodge numbers and compare them with the complete lists for toric hypersurface. Most of the new Hodge numbers lie near the lower "boundary region" at $h^{11} = 1$ and appear from a starting polyhedron in the N-lattice with less than 20 points. In particular, we doubled the number of known spectra with $h^{11} = 1$. Alltogether we found 87 pairs of new Hodge numbers not contained in the complete list of toric hypersurfaces [8]. They are listed in Table 2 and discussed in Section 6.2.

The paper is organized as follows. In Section 2 we recall some facts about TG, mainly to set up our notation. We will use the approach of the homogeneous coordinate ring, as introduced by Cox [22]. Some basic facts on the combinatorial data of nef partitions [23] and a new criterion for the nef property (Proposition 3.2), which was used in the computations, can be found in Section 3. In Section 4 we recall how to compute a Gorenstein cone from a nef partition [14,16] and Section 5 summarizes the polynomials defined in [16] to calculate the Hodge data for a Gorenstein cone arising from a nef partition. The formula used in our program can be found in Remark 5.8. In Section 6 we discuss a number of examples of complete intersections in $W\mathbb{P}^5$. We conclude with a discussion of our results, which will be posted at our web site [24] and some of which are listed in the appendix. A reader who is only interested in new results can take a look at Proposition 3.2 for a new criterion of a nef partition, Section 6.1 for comparing our results with codimension two Calabi–Yau manifolds in $W\mathbb{P}^5$ spaces [12], and Section 6.2 for new Hodge numbers.

2. TG and complete intersections

TG is a generalization of projective geometry where the gluing data of an algebraic variety are encoded in a fan Σ of convex rational cones. Often, the fan is given in terms of (the cones over the faces of) a polytope $\bar{\Delta}$ whose vertices lie on some lattice N [25,26]. A very useful way of defining these spaces is to introduce homogeneous coordinates z_i for all generators $v_i \in N$ ($i = 1, \dots, n$) of the one-dimensional cones in Σ (e.g. the vertices of $\bar{\Delta}$) and to consider the quotient of $\mathbb{C}^n - Z$ by identifications:

$$(z_1, \dots, z_n) \sim (\lambda^{q_1^{(I)}} z_1, \dots, \lambda^{q_n^{(I)}} z_n), \qquad \sum q_i^{(I)} v_i = 0, \quad I = 1, \dots, n-d,$$

where the scaling weights $q_i^{(I)}$ describe all linear relations among the generators v_i and $d = \dim(N)$ the dimension of the resulting toric variety $\mathbb{P}_\Sigma$ [22,27,28]. In the special case $n = d + 1$ of a weighted projective space the exceptional set Z, which is determined in terms of the fan, only consists of the origin $z_i = 0$.

Ample line bundles on $\mathbb{P}_\Sigma$ correspond to polytopes Δ in the dual space $M = \mathrm{Hom}(\mathbb{Z}, N)$ [25]. Toric varieties found their way into string theory when Batyrev [7] showed that the generic section of the line bundle corresponding to Δ defines a Calabi–Yau hypersurface in

$\mathbb{P}_\Sigma$ if $\bar{\Delta}$ is equal to the dual

$$\Delta^* = \{x \in N_\mathbb{R} | \langle y, x\rangle \geq -1 \quad \forall\, y \in \Delta \subset M_\mathbb{R}\}$$

of Δ, where $N_\mathbb{R}$ is the real extension of the lattice N. A lattice polytope Δ whose dual Δ^* is also a lattice polytope is called reflexive. A necessary condition for this is that the origin is the unique interior lattice point of Δ. Moreover, it turned out that the family of CY hypersurfaces in $\mathbb{P}_{\Sigma(\Delta)}$ that is defined by Δ provides the mirror family to the family of CY varieties that are based on $\bar{\Delta} = \Delta^*$ in the sense that the Hodge numbers $h^{p,q}$ and $h^{d-p,q}$ are exchanged [7]. At that time it had just become clear that hypersurfaces in weighted projective spaces are close to but not exactly mirror symmetric [5,6]. This is true even if orbifolds and discrete torsion are included [29,30], which do help in the situation where the Berglund–Hübsch [31,32] construction applies [33]. Beyond the construction of the missing mirror manifolds, however, Batyrev's results introduced to the physicist's community beautiful and extremely useful new techniques, which later turned out also to apply to the analysis of fibration structures that are important in string dualities [34–37]. In TG CY fibrations manifest themselves as reflexive sections or projections of the polytopes Δ^* and Δ, respectively [21,38].

3. The nef partitions

In the case of a hypersurface, the supporting polyhedron of the generic section of an ample line bundle on $\mathbb{P}_\Sigma$ must be reflexive in order to get a Calabi–Yau hypersurface in $\mathbb{P}_\Sigma$. To generalize this condition to the case of codimension $r \geq 1$, i.e. to ensure that the intersection of r hypersurfaces is a Calabi–Yau manifold, the reflexive polytope $\Delta \subset M_\mathbb{R}$ must fulfil the so called nef condition [23]. In this section, we will shortly discuss the combinatorial properties of nef partitions and give a new criterion for a reflexive polytope to decompose into a nef partition, Proposition 3.2, which can be used to calculate these partitions in a simple way, as described in Remark 3.4.

Let $\Delta \subset M_\mathbb{R}$ be a reflexive polytope and $\Delta^* \subset N_\mathbb{R}$ it's dual. From now on we denote by Δ^v the *set of vertices* of a polytope Δ. Let $E := \Delta^{*v}$ be the set of vertices of Δ^*. We define the d-dimensional complete *fan* $\Sigma[\Delta^*]$ as the union of the zero-dimensional cone $\{0\}$ together with the set of all cones:

$$C[F] = \{0\} \cup \{z \in N_\mathbb{R} : \lambda z \in F \text{ for some } \lambda \in \mathbb{R}_>\}$$

that support faces F of Δ^*. Assume that there exists a representation of $E = E_1 \cup \cdots \cup E_r$ as the union of disjoint subsets $E_1, \ldots, E_r$ and integral convex $\Sigma[\Delta^*]$ piecewise linear support functions $\varphi_i : N_\mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, r)$—such that

$$\varphi_i(e) = \begin{cases} 1 & \text{if } e \in E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Each φ_i corresponds to a line bundle L_i that defines a supporting polyhedron Δ_i for the global sections:

$$\Delta_i = \{\hat{z} \in M_\mathbb{R} : \langle \hat{z}, z\rangle \geq -\varphi_i(z)\, \forall\, z \in N_\mathbb{R}\}.$$

φ_i defines L_i in the following way. For each cone of maximal dimension C there is a $m_C \in M$ such that $\varphi_i|_C = m_C|_C$, where the m_C have to coincide on the intersection of two cones. Since the fan is complete, φ_i is determined uniquely by the set $\{m_C\}$. The line bundle L_i is then given by the data $(U_C, \chi(m_C))$, where $\{U_C\}$ is an open covering of the toric variety with open sets corresponding to the cones of maximal dimension and $\chi(m_C)$ can be regarded as a monomial x^{m_C}. The important point is that the transition functions $\chi(m_{\bar{C}} - m_C)$ arising from this construction are regular on the intersection of the corresponding open sets (for details see [25,26]).

Conversely, each function φ_i is uniquely defined by the polyhedron Δ_i. A *Calabi–Yau complete intersection* is then determined by the intersection of the closure of r hypersurfaces, each corresponding to a global section of a line bundle L_i [14,16].

Definition 3.1. If there exists a reflexive polytope Δ and r functions $\varphi_1, \dots, \varphi_r$ as defined above, we call the data

$$\Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\}$$

a *nef partition*.

Equivalent to $\Pi(\Delta) := \{\Delta_1, \dots, \Delta_r\}$ being a nef partition is that any two Δ_i only have $\{0\}$ as a common point and that Δ can be written as the Minkowski sum $\Delta_1 + \cdots + \Delta_r = \Delta$, as shown by the following proposition.

Proposition 3.2. *$\Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\}$ is a nef partition if and only if Δ is the Minkowski sum of r rational polyhedra $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\Delta_i \cap \Delta_j = \{0\}\, \forall\, i \neq j$.*

Proof. $\Rightarrow$: Assume that Δ can be written as the Minkowski sum of r rational polyhedra $\Delta = \Delta_1 + \cdots + \Delta_r$ with $\Delta_i \cap \Delta_j = \{0\}\, \forall\, i \neq j$. Define r functions $\varphi_i : N_{\mathbb{R}} \to \mathbb{R}$ as

$$\varphi_i(z) = -\min_{\hat{z} \in \Delta_i} \langle \hat{z}, z \rangle \quad \forall\, z \in N_{\mathbb{R}}.$$

- The φ_i are linear on cones of $\Sigma[\Delta^*]$. It is sufficient to consider restrictions of the φ_i to cones of maximal dimension $C[F]$, where

  $$F = \Delta^* \cap \{z \in N_{\mathbb{R}} : \langle \hat{e}, z \rangle = -1\}$$

  is a facet of Δ^* corresponding to a vertex $\hat{e} \in \Delta^v$. Now let $\hat{e} = \hat{e}_1 + \cdots + \hat{e}_i + \cdots + \hat{e}_r$, where $\hat{e}_i \in \Delta_i^v$ denotes a vertex of Δ_i $(i = 1, \dots, r)$. If we take another vertex $\hat{e}'_i \neq \hat{e}_i \in \Delta_i^v$, then the sum $\hat{e}' = \hat{e}_1 + \cdots + \hat{e}'_i + \cdots + \hat{e}_r$ denotes another vertex of Δ. Clearly, $\langle \hat{e}, z \rangle \leq \langle \hat{e}', z \rangle\, \forall\, z \in C[F]$, i.e. $\langle \hat{e}_i, z \rangle \leq \langle \hat{e}'_i, z \rangle\, \forall\, z \in C[F]$. Hence $\varphi_i(z) = -\langle \hat{e}_i, z \rangle\, \forall\, z \in C[F]$.
- Convexity of all φ_i follows immediately from their definition.
- $\varphi_i(e) \in \{0, 1\}\, \forall\, e \in E$, $i = 1, \dots, r$. For every function φ_i we observe that $0 \in \Delta_i$ implies $\varphi_i \geq 0$ and $\Delta_i \subseteq \Delta$ implies $\varphi_i(e) \leq 1\, \forall\, e \in E$.
- $\varphi_i(e) = 1 \Rightarrow \varphi_j(e) = 0\, \forall\, j \neq i$. Assume $\varphi_i(e) = \varphi_j(e) = 1$ for $i \neq j \Rightarrow \exists \hat{z}_i \in \Delta_i$, $\hat{z}_j \in \Delta_j : \langle \hat{z}_i, e \rangle = \langle \hat{z}_j, e \rangle = -1 \Rightarrow \exists \hat{z} = \hat{z}_i + \hat{z}_j \in \Delta$ with $\langle \hat{z}, e \rangle = -2$. This contradicts Δ^* being dual to Δ.

- $\forall\, e \in E \exists i \in \{1, \dots, r\}$ with $\varphi_i(e) = 1$. Assume $\exists e \in E : \varphi_i(e) = 0\, \forall\, i = 1, \dots, r$. By duality of Δ and $\Delta^* \exists \hat{z} \in \Delta : \langle \hat{z}, e\rangle = -1$, where $\hat{z}$ is contained in the facet dual to e. Now $\hat{z} = \hat{z}_1 + \cdots + \hat{z}_r$ with $\hat{z}_i \in \Delta_I\, \forall\, i = 1, \dots, r. \Rightarrow \exists \hat{z}_k \in \Delta_k$ with $\langle \hat{z}_k, e\rangle < 0$. This contradicts $\varphi_i(e) = 0\, \forall\, i = 1, \dots, r$.

$\Leftarrow$: Follows from

$$\Delta = \{\hat{z} \in M_{\mathbb{R}} : \langle \hat{z}, z\rangle \geq -\varphi(z)\, \forall\, z \in N_{\mathbb{R}}\},$$

where $\varphi = \sum_i \varphi_i$ $(i = 1, \dots, r)$. □

It can be shown that every nef partition of a reflexive polytope Δ gives a dual nef partition of a reflexive polytope ∇, which turns out to be an involution on the set of reflexive polytopes with nef partitions.

Remark 3.3 ([23]). Let $\Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\}$ be a nef partition and denote by $E = E_1 \cup \cdots \cup E_r$ the set of vertices Δ^{*v}. Define r rational polyhedra $\nabla_i \subset N_{\mathbb{R}}$ $(i = 1, \dots, r)$ as

$$\nabla_i = \mathrm{Conv}(E_i \cup \{0\}).$$

Then there is the following relation between Δ_i and ∇_j $(i, j = 1, \dots, r)$:

$$\langle \Delta_i, \nabla_j\rangle = \begin{cases} \geq -1 & \text{if } i = j, \\ \geq 0 & \text{otherwise} \end{cases}$$

and the ∇_i are maximal with that property. In particular $\nabla = \nabla_1 + \cdots + \nabla_r$ is a reflexive polyhedron with a nef partition $\Pi(\nabla) = \{\nabla_1, \dots, \nabla_r\}$, and there is a natural involution on the set of reflexive polyhedra with nef partitions:

$$\iota : \Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\} \mapsto \Pi(\nabla) = \{\nabla_1, \dots, \nabla_r\}.$$

Remark 3.4. The following procedure can be used to find all nef partitions of a reflexive polyhedron $\Delta \subset M_{\mathbb{R}}$:

- First calculate $\Delta^* \subset N_{\mathbb{R}}$.
- Take disjoint unions $E = E_1 \cup \cdots \cup E_r$ of vertices of Δ^*.
- Check if $\nabla = \nabla_1 + \cdots + \nabla_r$ with $\nabla_i = \mathrm{Conv}(E_i \cup \{0\})$ is reflexive and $\nabla_i \cap \nabla_j = \{0\}\, \forall\, i \neq j$.

4. Gorenstein cones

The (string theoretic) Hodge numbers of a Calabi–Yau manifold corresponding to a nef partition are the coefficients of the E-polynomial:

$$E_{\mathrm{st}}(V; u, v) = \sum (-1)^{p+q} h_{\mathrm{st}}^{p,q} u^p v^q,$$

which can be computed from a higher-dimensional Gorenstein cone [16] that is constructed using the data of a nef partition [14,16]. In this section we will give the definition of a Gorenstein cone and recall its construction starting with a nef partition.

A rational cone $C \subset M_{\mathbb{R}}$ is called *Gorenstein* if there exists a point $n \in N$ in the dual lattice such that $\langle v, n\rangle = 1$ for all generators of the semigroup $C \cap M$. Given a nef partition $\Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\}$, we can construct such a cone. First we go to a larger space and extend the canonical pairing. Let $\mathbb{Z}^r$ be the standard r-dimensional lattice and $\mathbb{R}^r$ its real scalar extension. We put $\bar{N} = \mathbb{Z}^r \oplus N$, $\bar{d} = d + r$ and $\bar{M} = \mathrm{Hom}(\bar{N}, \mathbb{Z})$. We extend the canonical $\mathbb{Z}$-bilinear pairing $\langle *, *\rangle : M \times N \to \mathbb{Z}$ to a pairing between $\bar{M}$ and $\bar{N} = \mathbb{Z}^r \oplus N$ by the formula

$$\langle (a_1, \dots, a_r, m), (b_1, \dots, b_r, n)\rangle = \sum_{i=1}^{r} a_i b_i + \langle m, n\rangle.$$

The real scalar extensions of $\bar{N}$ and $\bar{M}$ are denoted by $\bar{N}_{\mathbb{R}}$ and $\bar{M}_{\mathbb{R}}$, respectively, with the corresponding $\mathbb{R}$-bilinear pairing $\langle *, *\rangle$: $\bar{M}_{\mathbb{R}} \times \bar{N}_{\mathbb{R}} \to \mathbb{R}$.

Definition 4.1. For a nef partition $\Pi(\Delta) = \{\Delta_1, \dots, \Delta_r\}$ we construct a $\bar{d}$-dimensional Gorenstein cone $C_\Delta \subset \bar{M}_{\mathbb{R}}$

$$C_\Delta = \{(\lambda_1, \dots, \lambda_r, \lambda_1 \hat{z}_1 + \cdots + \lambda_r \hat{z}_r) \in \bar{M}_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq}, \hat{z}_i \in \Delta_i,\ i = 1, \dots, r\}$$

with $n_\Delta \in \bar{N}$ uniquely defined by the conditions

$$\langle \hat{z}, n_\Delta\rangle = 0 \quad \forall\, \hat{z} \in M_{\mathbb{R}} \subset \bar{M}_{\mathbb{R}}, \qquad \langle \hat{e}_i, n_\Delta\rangle = 1 \quad \text{for } i = 1, \dots, r,$$

where $\{\hat{e}_1, \dots, \hat{e}_r\}$ is the standard basis of $\mathbb{Z}^r \subset \bar{M}$.

Note that all generators of $C_\Delta \cap \bar{M}$ lie on the hyperplane $\langle \hat{z}, n_\Delta\rangle = 1$. They span the $d-1$-dimensional supporting polyhedron

$$K_\Delta = \{\hat{z} \in C_\Delta : \langle \hat{z}, n_\Delta\rangle = 1\}$$

of C. Since $K_\Delta \cap \bar{M}$ has no interior point, we get

$$K_\Delta \cap \bar{M} = \bigcup_{i=1,\dots,r} (\hat{e}_i \times \Delta_i) \cap \bar{M}.$$

Remark 4.2 ([14]). Let $\Pi(\nabla) = \{\nabla_1, \dots, \nabla_r\}$ be the dual nef partition. Then the Gorenstein cone

$$C_\nabla = \{(\mu_1, \dots, \mu_r, \mu_1 z_1 + \cdots + \mu_r z_r) \in \bar{N}_{\mathbb{R}} : \mu_i \in \mathbb{R}_{\geq}, z_i \in \nabla_i,\ i = 1, \dots, r\}$$

is dual to C_Δ defined in Definition 4.1. Note, however, that K_Δ is not dual to K_∇!

5. Combinatorial polynomials of Eulerian posets

Batyrev and Borisov [16] gave an explicit formula for the string-theoretic E-polynomial for a Calabi–Yau complete intersection V in a Gorenstein toric Fano variety. This polynomial depends only on the combinatorial data of the corresponding Gorenstein cone. We will give some basic definitions of combinatorial polynomials on Eulerian Posets, which are used to compute the E-polynomial, and formulate it in a way which can be used for the explicit calculation of the Hodge numbers.

Let P be an *Eulerian Poset*, i.e. a finite partially ordered set with unique minimal element $\hat{0}$, maximal element $\hat{1}$ and the same length d of every maximal chain of P. For any $x \leq y \in P$, define the *interval* $I = [x, y]$ as

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

In particular, we have $P = [\hat{0}, \hat{1}]$. Define the *rank function* $\rho : P \to \{0, \dots, d\}$ on P by setting $\rho(x)$ equal to the length of the interval $[\hat{0}, x]$. Note that for any Eulerian Poset P, every interval $I = [x, y]$ is again an Eulerian Poset with rank function $\rho(z) - \rho(x)\, \forall z \in I$. If an Eulerian Poset has rank d, then the *dual Poset* P^* is also an Eulerian Poset with rank function $\rho^* = d - \rho$.

Example 5.1. Let $C \in N_{\mathbb{R}}$ be a d-dimensional cone with its dual $C^* \in M_{\mathbb{R}}$. There is a canonical bijective correspondence $F \leftrightarrow F^*$ between faces $F \subseteq C^*$ and $F^* \subseteq C^*$ with $\dim F + \dim F^* = d$ [25]:

$$F \mapsto F^* = \{z \in C^* : \langle \hat{z}, z\rangle = 0\, \forall z \in F\},$$

which reverses the inclusion relation between faces. We denote the faces of C by indices x and define the poset $P = [\hat{0}, \hat{1}]$ as the poset of all faces $C_x \subseteq C$ with maximal element C and minimal element $\{0\}$ and rank function $\rho(x) = \dim(C_x)\, \forall x \in P$. The dual poset P^* can be identified with the poset of faces $C_x^* \subseteq C^*$ of the dual cone C^* with rank function $\rho^*(x^*) = \dim(C_x^*)\, \forall x^* \in P^*$.

Definition 5.2. Let P be an Eulerian Poset of rank d as above. Define the *polynomial* $B(P; u, v) \in \mathbb{Z}[u, v]$ by the following rules [16,39]:

- $B(P; u, v) = 1$ if $d = 0$;
- The degree of $B(P; u, v)$ with respect to v is less than $d/2$;
- $\sum_{\hat{0} \leq x \leq \hat{1}} B([\hat{0}, x]; u^{-1}, v^{-1})(uv)^{\rho(x)}(v-u)^{d-\rho(x)} = \sum_{\hat{0} \leq x \leq \hat{1}} B([x, \hat{1}]; u, v)(uv-1)^{\rho(x)}$.

Let us consider how we can construct the B-polynomial for an interval $I = [x, y] \subseteq P$ with $d = \rho(y) - \rho(x)$. Suppose we know the B-polynomials $B(\tilde{I}; u, v)$ for all sub-intervals $\tilde{I} = [\tilde{x}, \tilde{y}] \subset I$. Then we know all terms of the relation formula for the B-polynomials in 5.2 except for $B(I; u, v)$ on the right hand side and $B(I; u^{-1}, v^{-1})(uv)^d$ on the left hand side. Because the v-degree of $B(I; u, v)$ is less than $d/2$, the possible degrees of monomials with respect to v in $B(I; u, v)$ and $B(I; u^{-1}, v^{-1})(uv)^d$ do not coincide and we can calculate $B(I; u, v)$. So if we have to compute $B(P; u, v)$, we first have to calculate the B-polynomials for all intervals with rank 0 (which are per definition 1), then those intervals with rank 1, etc.

Remark 5.3. Let P be an Eulerian Poset of rank d, P^* be the dual. Then the polynomial defined in 5.2 satisfies

$$B(P; u, v) = (-u)^d B(P^*; u^{-1}, v).$$

Definition 5.4. Let P be the Eulerian Poset corresponding to the Gorenstein cone $C = C_\Delta \subset \bar{M}_\mathbb{R}$ from Definition 4.1. Define two functions on the set of faces of C by

$$S(C_x, t) = (1-t)^{\rho(x)} \sum_{m \in C_x \cap \bar{M}} t^{\deg(m)}, \qquad T(C_x, t) = (1-t)^{\rho(x)} \sum_{m \in \mathrm{Int}(C_x) \cap \bar{M}} t^{\deg(m)},$$

where $\mathrm{Int}(C_x)$ denotes the relative interior of $C_x \subseteq C$ and $\deg(m) = \langle m, n_\Delta \rangle$.

The following statement is a consequence of the Serre duality [40].

Proposition 5.5. *For the Gorenstein cone $C = C_\Delta \subset \bar{M}_\mathbb{R}$ the functions S and T are polynomials*: (C_x, t), $T(C_x, t) \in \mathbb{Z}[t]$, *and they satisfy the relation*

$$S(C_x, t) = t^{\rho(x)} T(C_x, t^{-1}).$$

Remark 5.6. For $S = \sum_i a_i t^i$ and $T = \sum_i b_i t^i$ as defined above 5.5 implies that

$$a_0 + a_1 t + \cdots + a_n t^n = b_0 t^n + b_1 t^{n-1} + \cdots + b_{n-1} t + b_n,$$

where $n = \dim C_x$ and we get the relations

$$a_i = b_{n-i} (i = 1, \ldots, n)$$

for the coefficients of S and T. Since $a_0 = 1$ and $b_0 = 0$, the leading coefficients are determined to be $a_n = 0$ and $b_n = 1$. So it is sufficient to calculate $|C_x \cap m \cdot K_\Delta|$ and $|\mathrm{Int}(C_x \cap m \cdot K_\Delta)|$ for $m = 0, \ldots, [\dim(C_x)/2]$ and to use the fact that $a_i = b_{n-i}$ for $i > \dim((C_x)/2)$.

Batyrev and Borisov [16] showed in their paper that the *string-theoretic E-polynomial* of a nef partition can be calculated from the data of the corresponding Gorenstein cone.

Proposition 5.7 ([16]). *Let $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ be a nef partition and $C = C_\Delta \subset \bar{M}_\mathbb{R}$ be the $\bar{d}$-dimensional reflexive Gorenstein cone defined in* 4.1 (*with dual cone $C^* = C_\nabla \subset \bar{N}_\mathbb{R}$*). *Denote by P the poset of faces $C_x \subseteq C$* (*see* Example 5.1). *Then the string-theoretic E-polynomial is given by*

$$E_{\mathrm{st}}(V; u, v) = \sum_{I=[x,y] \subseteq P} \frac{(-1)^{\rho(y)}}{(uv)^r} (v-u)^{\rho(x)} B(I^*; u, v)(uv-1)^{\bar{d}-\rho(y)} A_{(x,y)}(u, v)$$

with

$$A_{(x,y)}(u, v) = \sum_{(m,n) \in \mathrm{Int}(C_x) \cap \bar{M} \times \mathrm{Int}(C_y^*) \cap \bar{N}} \left(\frac{u}{v}\right)^{\deg(m)} \left(\frac{1}{uv}\right)^{\deg(n)}.$$

The dual partition $\Pi(\nabla) = \{\nabla_1, \dots, \nabla_r\}$ corresponds to the Calabi–Yau complete intersection W and (V, W) is a mirror pair of (singular) Calabi–Yau varieties, at least in the sense that

$$E_{\mathrm{st}}(V; u, v) = (-u)^{d-r} E_{\mathrm{st}}\left(W; \frac{1}{u}, v\right)$$

or equivalently $h_{\mathrm{st}}^{p,q}(V) = h_{\mathrm{st}}^{n-p,q}(W)$ *for* $0 \le p, q \le n = \dim(V) = \dim(W)$.

Remark 5.8. Using the duality 5.3 for the B-polynomials and Definition 5.4 with relation 5.5 between the S- and T-polynomials, we can write the E-polynomial as

$$E_{\mathrm{st}}(V; u, v) = \sum_{I=[x,y]\subseteq P} \frac{(-1)^{\rho(x)} u^{\rho(y)}}{(uv)^r} S\left(C_x, \frac{v}{u}\right) S(C_y^*, uv) B(I; u^{-1}, v).$$

This equation can be used for explicit calculations.

6. Results

Using the formula for the E-polynomial 5.8 we are now able to construct Calabi–Yau complete intersections starting with a reflexive polytope $\Delta \subset M_{\mathbb{R}}$ (or $\Delta^* \subset N_{\mathbb{R}}$). Our first task is to compare the toric construction to a list of complete intersections in weighted projective spaces, which was produced by Klemm [12]. Then we construct a larger number of nef partitions for different classes of five-dimensional reflexive polytopes and compare the Hodge data with the complete results that are available for toric hypersurfaces [9,24].

6.1. Comparison with weighted projective space

In order to identify complete intersections in $W\mathbb{P}^5$ as special cases of the toric construction it is natural to start with the Newton polyhedron and compare the Hodge data for various nef partitions. In what follows we will analyze some examples from Klemm's list [12] and discuss the different situations that can occur.

In the simplest case the Newton polyhedron $\Delta(d)$ corresponding to degree (d_1, d_2) equations with $d = d_1 + d_2$ is reflexive and the Hodge numbers of a nef partition $\Pi(\Delta(d)) = \{\tilde{\Delta}_1, \tilde{\Delta}_2\}$ agree with those in [12]. This works already for the first example of degree (4, 2) equations in $\mathbb{P}_{1,1,1,1,1,1}$. In general, $\Delta(d_1 + d_2)$ may differ from the Minkowski sum $\Delta(d_1) + \Delta(d_2)$ and none of the two plytopes has to be reflexive. In many cases we find a simple modification of these polytopes that makes the Hodge data agree.

- Already in the second example of this list the Newton polyhedron $\Delta(7)$ for the weight system of $W\mathbb{P}_{1,1,1,1,1,2}$ is not reflexive. It is, however, possible to reduce $\Delta(7)$ to a reflexive polyhedron Δ by omitting five points, so that its dual provides a toric resolution of singularities of the weighted projective space. Indeed, the Hodge data for $(d_1, d_2) =$ (3, 4) matches for one nef partition of the resulting polytope.

- Another possibility is that the Newton polyhedron is reflexive, but the Hodge numbers do not agree. In such a case we can compute the Minkowski sum $\tilde{\Delta} = \Delta(d_1) + \Delta(d_2)$ and check if it is reflexive and gives the right Hodge numbers. This works, e.g. for degrees $(d_1, d_2) = (5, 3)$ in case of the weight system for $W\mathbb{P}_{1,1,1,1,2,2}$.

There are still some examples where we are not able to reproduce the Hodge data. For example, in case of the weight system for $W\mathbb{P}_{1,1,1,1,2,3}$ and degrees $(d_1, d_2) = (5, 4)$, neither $\Delta(9)$ (which has 575 points) nor the Minkowski sum $\Delta(5) + \Delta(4)$ (with only 211 points) is reflexive. The largest reflexive subpolytope of $\Delta(9)$ has 570 points, but it's nef partitions do not yield the right Hodge numbers. Omitting up to 30 points we find another 21 reflexive polyhedra, but none of their nef partitions yields $h^{11} = 2$ and $h^{12} = 84$. We thus found no candidate for a toric description and a more detailed analysis of the geometry would be required to check if a toric description exists.

6.2. *New Hodge numbers*

Of course, one of our main interests is to find new Hodge data. In [9,24] the complete set of 30 108 pairs of Hodge numbers corresponding to hypersurfaces in toric varieties has been constructed (the number is 15,122 if we count those with $h^{11} \geq h^{12}$ because 136 are self-dual). Picking out only the new Hodge data from the list of 2387 pairs arising from weighted $\mathbb{P}^5$ [12], i.e. those not contained in [24], there remain only 15 new data, which we list in Table 1. (Note that this class is not mirror symmetric.)

Using the toric construction, we started with reflexive polyhedra that are described by single or combined weight systems, as they were constructed systematically for Calabi–Yau 4-fold [11,21,24], and from Minkowski sums of Newton polytopes that arise in the context of weighted projective spaces. In this way we found 16 (with mirror duality 32) pairs of new Hodge numbers. They are listed in Appendix A, together with a detailed information about the starting polyhedron. Most of them lie in the lower boundary region $h^{11} \leq 6$ which is less covered by the "background" of toric hypersurfaces. It is remarkable that almost every pair of new Hodge numbers corresponds to a starting polyhedron $\Delta^* \cap N$ in the N-lattice with less than 20 points. Thus, to get a more complete result for new spectra in that range, we used the program package that was written for the classification of reflexive polyhedra [24,41] to construct a fairly complete set of reflexive polyhedra with up to 10 points (they were all found as subpolytopes of some 10 000 polyhedra with up to 40 points originating from transversal weight systems [21]). Indeed, using these polytopes for $\Delta^* \cap N$, we found

Table 1
New Hodge numbers in [12], as compared to toric hypersurfaces ($R = x$ means that we found the same Hodge data for nef partitions)

h^{11}	h^{21}	R	h^{11}	h^{21}	R	h^{11}	h^{21}	R
1	61	x	2	62	x	7	26	
1	73	x	2	68	x	8	20	
1	79	x	3	47	x	12	12	
1	89	x	3	55	x	13	13	x
1	129	x	3	61	x	17	11	

Table 2
New Hodge numbers with toric CICYs.

h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}	h^{11}	h^{21}
1	25	2	60	3	24	3	52	4	16	4	53
1	37	2	62	3	27	3	53	4	22	4	75
1	61	2	64	3	29	3	54	4	24	4	83
1	73	2	66	3	31	3	55	4	26	5	25
1	79	2	68	3	33	3	56	4	30	5	27
1	89	2	70	3	35	3	58	4	31	5	102
1	129	2	72	3	37	3	60	4	32	6	20
2	30	2	76	3	39	3	61	4	33	6	24
2	36	2	77	3	41	3	62	4	38	7	22
2	44	2	78	3	42	3	64	4	39	8	14
2	50	2	80	3	44	3	68	4	41	13	13
2	54	2	82	3	47	3	70	4	43	13	15
2	56	2	100	3	48	3	80	4	45		
2	58	2	112	3	49	3	101	4	47		
2	59	3	23	3	50	3	113	4	51		

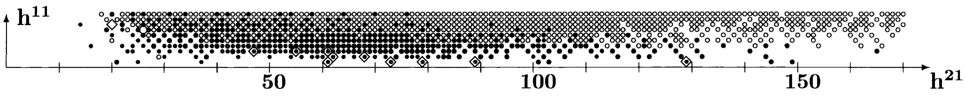

Fig. 1. Toric CICYs (•) and CICYs in $W\mathbb{P}^5$ (◇) in the background of toric hypersurfaces (○) with $h^{11} \le 10$ $h^{21} \le 170$.

87 pairs of Hodge numbers not contained in [24]. They are listed in Table 2 and are shown in Fig. 1 in the background of toric hypersurfaces and CICYs in $W\mathbb{P}^5$ in the range of $1 \le h^{11} \le 10$ and $1 \le h^{21} \le 170$.

The advantage of this strategy is that it is easy to get a rather complete list of reflexive polytopes with a small number of points, which at the same time have a high probability for the existence of nef partitions and whose Hodge data are outside the range that is already completely covered by hypersurfaces. Moreover, this class dissipates less time in computing the nef partitions because of their small number of vertices. Pursuing this strategy, a further step will be to increase the codimension by one and to construct complete intersections using six-dimensional starting polytopes with a small number of points.

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Appendix A. Hodge data

New Hodge numbers of toric CICYs using combined weight systems

d_1	$w_{1_1},\ldots,w_{n_1}$	d_2	$w_{1_2},\ldots,w_{n_2}$	h^{11}	h^{21}	$-\chi$	$\#\Delta\cap M$	$\#\Delta^v$	$\#\Delta^*\cap N$	$\#\Delta^{*^v}$
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	59	114	350	12	8	7
3	1 1 1 0 0 0 0	5	1 0 0 1 1 1 1	2	60	116	379	12	8	7
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	62	120	350	12	8	7
3	1 1 1 0 0 0 0	6	0 1 1 1 1 1 1	2	62	120	381	12	8	7
3	1 1 1 0 0 0 0	5	1 0 0 1 1 1 1	2	70	136	379	12	8	7
3	1 1 1 0 0 0 0	6	0 1 1 1 1 1 1	2	76	148	381	12	8	7
3	1 1 1 0 0 0 0	4	0 0 0 1 1 1 1	2	77	150	350	12	8	7
3	1 1 1 0 0 0 0	8	3 0 0 1 1 2 1	2	100	196	496	16	9	8
4	1 1 2 0 0 0 0	5	1 0 0 1 1 1 1	3	55	104	292	12	9	7
4	1 1 2 0 0 0 0	6	1 0 0 1 1 1 2	3	55	104	282	12	9	7
4	2 1 1 0 0 0 0	8	0 1 2 1 1 1 2	3	55	104	247	9	9	7
4	2 1 1 0 0 0 0	12	0 1 3 2 2 2 2	3	55	104	265	9	9	7
4	1 1 2 0 0 0 0	4	0 0 0 1 1 1 1	3	55	104	315	12	9	7
3	1 1 1 0 0 0 0	5	0 0 0 1 1 1 2	3	56	106	340	18	9	8
4	1 1 2 0 0 0 0	16	1 0 0 2 3 4 6	13	15	4	117	20	15	10

New Hodge numbers of toric CICYs using one weight system

d	$w_1,\ldots,w_n$	h^{11}	h^{21}	$-\chi$	$\#\Delta\cap M$	$\#\Delta^v$	$\#\Delta^*\cap N$	$\#\Delta^{*^v}$
12	1 1 2 2 3 3	1	61	120	407	6	7	6
6	1 1 1 1 1 1	1	73	144	462	6	7	6
8	1 1 1 1 2 2	1	73	144	483	6	7	6
6	1 1 1 1 1 1	1	89	176	462	6	7	6
12	1 2 2 2 2 3	2	62	120	321	6	8	6
9	1 1 1 2 2 2	2	68	132	434	12	8	7
10	1 1 2 2 2 2	2	68	132	378	6	8	6

New Hodge numbers of toric CICYs using the Minkowski sum

d	$w_1,\ldots,w_n$	d_1	d_2	h^{11}	h^{21}	$-\chi$	$\#\Delta\cap M$	$\#\Delta^v$	$\#\Delta^*\cap N$	$\#\Delta^{*^v}$
12	1 1 2 2 3 3	6	6	1	61	120	407	6	7	6
6	1 1 1 1 1 1	2	4	1	73	144	462	6	7	6
8	1 1 1 1 2 2	4	4	1	73	144	483	6	7	6
6	1 1 1 1 1 1	2	4	1	89	176	462	6	7	6
8	1 1 1 1 1 3	4	4	1	129	256	636	10	8	7
12	1 2 2 2 2 3	6	6	2	62	120	321	6	8	6
9	1 1 1 2 2 2	4	5	2	68	132	434	12	8	7
10	1 1 2 2 2 2	4	6	2	68	132	378	6	8	6
14	1 2 2 3 3 3	8	6	3	47	88	294	12	9	7
16	1 2 2 3 4 4	8	8	3	55	104	327	8	9	7

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